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1981 J. Phys. A: Math. Gen. 14 L389

(http://iopscience.iop.org/0305-4470/14/10/002)

J. Phys. A: Math. Gen. 14 (1981) L389–L391. Printed in Great Britain

LETTER TO THE EDITOR

The positivity of the Bondi mass

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Received 21 July 1981

Abstract. Two component spinor techniques, similar to those used by Witten, are used to express the Bondi momentum of an asymptotically flat space–time in the form of an integral over an asymptotically null, space-like hypersurface Σ. It is then shown that the Bondi mass is positive, given the existence of a Green function for the 'Witten equation' on Σ.

In a recent paper Witten (1981) has given a simple proof of the positivity of the ADM mass of an asymptotically flat space–time by means of a simple argument using Dirac spinors. Witten's proof of the positive energy theorem is not, however, quite correct as it stands as it uses an invalid three-dimensional truncation of the four-dimensional Gauss divergence theorem. By giving a new four-dimensional covariant expression for the ADM four-momentum, Nester (1981) has been able to avoid this difficulty and give a simple proof of the positivity of the ADM mass. We show here how similar techniques may be used in the case of the Bondi mass.

Let M be an asymptotically flat space–time with future null infinity $\mathscr{I}^+$. The Bondi–Sachs (Bondi *et al* 1962, Sachs 1962) four-momentum $P_a(S)$ of M is a four-vector function, defined on the space of all space-like cross sections (cuts) of $\mathscr{I}^+$, which lies in the Minkowski space of BMS translations T. If we let $T = \mathscr{S} \otimes \bar{\mathscr{S}}$ where $\mathscr{S}$ is the space of two-spinors, then, on using the Penrose abstract index notation (Penrose 1968), we may write

$$P_a = P_{AA'}$$

where the indices A and A' refer to $\mathscr{S}$ and $\bar{\mathscr{S}}$ respectively. In terms of the standard spin-coefficient notation based on a Bondi coordinate system $(u, r, \zeta, \bar{\zeta})$ and associated spinor dyad field (o_A, ι_A) (see, for example, Exton *et al* 1969), the Bondi momentum with respect to the origin cut $u = 0$ may be written as

$$P_{AA'} = -\tfrac{1}{2} \oint (\psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0) \mathrm{O}_A \mathrm{O}_{A'}\, \mathrm{d}\Omega \tag{1}$$

where the integral is performed over the $u = 0$ cut of $\mathscr{I}^+$. $\mathrm{O}_A(\zeta, \bar{\zeta})$ is a regular spinor valued function lying in $\mathscr{S}$ which has spin weight $\frac{1}{2}$ and which satisfies

$$\bar{\eth}\mathrm{O}_A = 0 \qquad \text{and} \qquad \mathrm{O}_A I^A = 1 \tag{2}$$

where

$$I_A = -\bar{\eth}\mathrm{O}_A. \tag{3}$$

The differential operators $\eth$ and $\bar{\eth}$ are the standard 'edth' operators of Newman and Penrose (1966).

We wish to show that the Bondi mass is positive in the sense that P_a is future pointing. This will be the case if and only if

$$P_{\mathbf{A}\mathbf{A}'}\lambda^{\mathbf{A}}\lambda^{\mathbf{A}'} \geqslant 0$$

for any arbitrary spinor $\lambda^{\mathbf{A}}$.

We first introduce an asymptotically null, space-like hypersurface Σ which is given asymptotically in Bondi coordinates by

$$0 = u - 1/r + O(r^{-2}). \tag{4}$$

This surface intersects $\mathscr{I}^+$ at the origin cut $u = 0$. Let λ_A be a spinor field on Σ which is arbitrary apart from the following asymptotic conditions:

$$\lambda_A o^A = \lambda_0 = \lambda_0^0(\zeta, \bar{\zeta}) + O(r^{-1}), \qquad \lambda_A \iota^A = \lambda_1 = \lambda_1^0(\zeta, \bar{\zeta}) + O(r^{-1}),$$

where

$$\eth\lambda_0^0 = 0 \quad \text{and} \quad \bar{\eth}\lambda_0^0 = -\lambda_1^0. \tag{6}$$

By virtue of these conditions, this spinor field determines a unique spinor $\lambda_{\mathbf{A}}$ in $\mathscr{S}$ according to

$$\lambda_{\mathbf{A}} = \lambda_1^0 O_{\mathbf{A}} - \lambda_0^0 I_{\mathbf{A}}. \tag{7}$$

We next introduce an antisymmetric tensor field F_{ab} on Σ given by

$$F_{ab} = \varepsilon_{A'B'}\phi_{AB} + \text{cc} \tag{8}$$

where

$$\phi_{AB} = \tfrac{1}{2}\lambda_{C'}\nabla^{C'}_{(A}\lambda_{B)} - \tfrac{1}{2}\lambda_{(A}\nabla^{C'}_{B)}\lambda_{C'} \tag{9}$$

and cc stands for complex conjugate. After a relatively simple calculation which makes use of equations (5), (6) and (7) plus the asymptotic form of the spin coefficients, one can show that

$$I = \lim_{r\to\infty} \oint F_{ab} l^a n^b \, d\Omega = -P_{\mathbf{A}\mathbf{A}'}\lambda^{\mathbf{A}}\lambda^{\mathbf{A}'} \tag{10}$$

where $l_a = o_A o_{A'}$, $n_a = \iota_A \iota_{A'}$. Thus, in order to show the positivity of the Bondi mass, it is sufficient to show that $I \leqslant 0$.

By Gauss' theorem we have

$$I = 2^{-1/2} \int \nabla^b F_{ab} v^a \, dv \tag{11}$$

where v^a is normal to Σ and satisfies $v^a v_a = 2$, and dv is the intrinsic volume element of Σ. On using the irreducible spinor components of the Ricci identity

$$\nabla_{[c}\nabla_{d]}T_b = \tfrac{1}{2}R^a{}_{bcd}T_a \tag{12}$$

(this equation, incidentally, fixes the sign of our Riemann tensor), the integrand in (11) may be calculated to be

$$(\nabla^b F_{ab})v^a = \{3G_{ab}K^a v^b - [(2\nabla^A_{B'}\lambda_{C'}\nabla^{C'}_{(A}\lambda_{B)} - 2\nabla^A_{B'}\lambda_{(A}\nabla^{C'}_{B)}\lambda_{C'}) + \text{cc}]v^{BB'}\} \tag{13}$$

where $K^a = \lambda^A\lambda^{A'}$ and G_{ab} is the Einstein tensor. If we now restrict λ^A to satisfy the 'Witten equation'

$$D^A_{A'}\lambda_A = 0 \tag{14}$$

where $D_a = \nabla_a - \frac{1}{2}v_a(v^c\nabla_c)$ then equation (13) simplifies to

$$(\nabla^b F_{ab})v^a = 3G_{ab}K^a v^b - 4(D^A_{B'}\lambda_{C'} D^{C'}_B \lambda_A)v^{BB'}$$

which can be seen to be less than or equal to zero provided the energy condition

$$G_{ab}K^a v^b \leqslant 0$$

holds on Σ. Thus given the existence of a spinor satisfying (5), (6) and (14) we have shown that the Bondi momentum is future pointing and hence that the Bondi mass is positive.

It remains to show that we may define a spinor field λ_A which satisfies (5), (6) and (14). We may show the existence of such a spinor field by using arguments similar to those of Witten (1981) provided we can show the existence of a Green function, with the appropriate behaviour at infinity, for the differential operator $D^A_{A'}$. Since Σ is an asymptotically null hypersurface, $D^A_{A'}$ becomes non-elliptic as $r \to \infty$ and we are thus unable to use any of the standard results to show the existence of the Green function. However, given the existence of λ_A satisfying (5), (6) and (14), it is possible to determine an asymptotic expansion for λ_A in terms of $1/r^m$ and $\log r/r^m$ as $r \to \infty$, and we have done so explicitly as far as terms in $\log r/r^4$. While such an asymptotic expansion does not prove the existence of the Green function, it suggests that the Green function required does indeed exist.

We thank Dr K P Tod for a number of useful conversations and also for bringing to our attention Witten's proof of the positive energy theorem.

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